Exercise 11

Consider an electrical cable running along the x-axis that is not well insulated from ground, so that leakage occurs along its entire length. Let $V(x,t)$ and $I(x,t)$ denote the voltage and current at point x in the wire at time t. These functions are related to each other by the system

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI \quad \text{and} \quad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV,$$

where L is the inductance, R is the resistance, C is the capacitance, and G is the leakage to ground. Show that V and I each satisfy

$$\frac{\partial^2 u}{\partial x^2} = LC\frac{\partial^2 u}{\partial t^2} + (RC + LG)\frac{\partial u}{\partial t} + RGu,$$

which is called the **telegraph equation**.

Solution

Differentiate both sides of the equation on the left and right with respect to x and t, respectively.

$$\begin{cases} \dfrac{\partial V}{\partial x} = -L\dfrac{\partial I}{\partial t} - RI \\ \\ \dfrac{\partial I}{\partial x} = -C\dfrac{\partial V}{\partial t} - GV \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(-L\dfrac{\partial I}{\partial t} - RI\right) \\ \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial I}{\partial x}\right) = \dfrac{\partial}{\partial t}\left(-C\dfrac{\partial V}{\partial t} - GV\right) \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial^2 V}{\partial x^2} = -L\dfrac{\partial^2 I}{\partial x \partial t} - R\dfrac{\partial I}{\partial x} \\ \\ \dfrac{\partial^2 I}{\partial t \partial x} = -C\dfrac{\partial^2 V}{\partial t^2} - G\dfrac{\partial V}{\partial t} \end{cases}$$

The mixed partial derivatives are equal by Clairaut's theorem,

$$\frac{\partial^2 I}{\partial x \partial t} = \frac{\partial^2 I}{\partial t \partial x},$$

so substitute the formulas for I_{tx} and I_x into the one for V_{xx}.

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= -L\frac{\partial^2 I}{\partial x \partial t} - R\frac{\partial I}{\partial x} \\ &= -L\left(-C\frac{\partial^2 V}{\partial t^2} - G\frac{\partial V}{\partial t}\right) - R\left(-C\frac{\partial V}{\partial t} - GV\right) \\ &= LC\frac{\partial^2 V}{\partial t^2} + (RC + LG)\frac{\partial V}{\partial t} + RGV \end{aligned}$$

Therefore, V satisfies the telegraph equation. Now instead, differentiate both sides of the equation on the left and right with respect to t and x, respectively.

$$\begin{cases} \dfrac{\partial V}{\partial x} = -L\dfrac{\partial I}{\partial t} - RI \\ \\ \dfrac{\partial I}{\partial x} = -C\dfrac{\partial V}{\partial t} - GV \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial V}{\partial x}\right) = \dfrac{\partial}{\partial t}\left(-L\dfrac{\partial I}{\partial t} - RI\right) \\ \\ \dfrac{\partial}{\partial x}\left(\dfrac{\partial I}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(-C\dfrac{\partial V}{\partial t} - GV\right) \end{cases} \quad \rightarrow \quad \begin{cases} \dfrac{\partial^2 V}{\partial t \partial x} = -L\dfrac{\partial^2 I}{\partial t^2} - R\dfrac{\partial I}{\partial t} \\ \\ \dfrac{\partial^2 I}{\partial x^2} = -C\dfrac{\partial^2 V}{\partial x \partial t} - G\dfrac{\partial V}{\partial x} \end{cases}$$

The mixed partial derivatives are equal by Clairaut's theorem,

$$\frac{\partial^2 V}{\partial t \partial x} = \frac{\partial^2 V}{\partial x \partial t},$$

so substitute the formulas for V_{tx} and V_x into the one for I_{xx}.

$$\begin{aligned}\frac{\partial^2 I}{\partial x^2} &= -C\frac{\partial^2 V}{\partial x \partial t} - G\frac{\partial V}{\partial x} \\ &= -C\left(-L\frac{\partial^2 I}{\partial t^2} - R\frac{\partial I}{\partial t}\right) - G\left(-L\frac{\partial I}{\partial t} - RI\right) \\ &= LC\frac{\partial^2 I}{\partial t^2} + (RC + LG)\frac{\partial I}{\partial t} + RGI\end{aligned}$$

Therefore, I satisfies the telegraph equation.